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ON CUBE DIVISOR CORDIAL GRAPHS

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ABSTRACT

The present authors are motivated by two research articles "Divisor Cordial Graphs" by Varatharajan et al. and "Square Divisor Cordial Graphs" by Murugesan et al. We introduce the concept of cube divisor cordial labeling. A cube divisor cordial labeling of a graph G with vertex set V is a bijection f from V to $\{1, 2, \ldots, |V|\}$ such that an edge $e = uv$ is assigned the label 1 if $[f(u)]^3 \mid f(v)$ or $[f(v)]^3 \mid f(u)$ and the label 0 otherwise, then $|e_f(0) - e_f(1)| \leq 1$. A graph which admits a cube divisor cordial labeling is called a cube divisor cordial graph. In this paper we investigate cube divisor cordial labeling of complete graph, star graph, complete bipartite graphs $(K_{2,n}$ & $K_{3,n})$, bistar $B_{n,n}$ and restricted square graph of $B_{n,n}$.

KEYWORDS: Cube Divisor Cordial Labeling; Complete Graph; Star Graph; Complete Bipartite Graph

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INTRODUCTION

Throughout this work, by a graph we mean finite, connected, undirected, simple graph $G = (V(G), E(G))$ of order $|V(G)|$ and size $|E(G)|$.

Definition 1.1: A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain condition(s).

The most recent findings on various graph labeling techniques can be found in Gallian $[2]$.

Notation 1.2: $e_f(i)$ = Number of edges with label i; $i = 0, 1$.

Definition 1.3: Let $G = (V(G), E(G))$ be a simple graph and $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ be a bijection. For each edge $e = uv$, assign the label 1 if $f(u) \mid f(v)$ or $f(v) \mid f(u)$ and the label 0 otherwise. The function f is called a *divisor cordial labeling* if $|e_f(0) - e_f(1)| \leq 1$.

A graph which admits a divisor cordial labeling is called a *divisor cordial graph*.

The concept of divisor cordial labeling was introduced by Varatharajan et al. [8] and they proved the following results

- The star graph $K_{1,n}$ is divisor cordial.

- The complete bipartite graphs $K_{2,n}$ and $K_{3,n}$ are divisor cordial.
- The complete graph K_n is not divisor cordial for $n \geq 7$.
- The bistar $B_{m,n}$ $(m \leq n)$ is divisor cordial.

Definition 1.4: Let $G = (V(G), E(G))$ be a simple graph and $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ be a bijection. For each edge $e = uv$ assign the label 1 if $[f(u)]^2 \mid f(v)$ or $[f(v)]^2 \mid f(u)$ and the label 0 otherwise. The function f is called a *square divisor cordial labeling* if $|e_f(0) - e_f(1)| \leq 1$.

A graph which admits a square divisor cordial labeling is called a *square divisor cordial graph*.

The concept of square divisor cordial labeling was introduced by Murugesan et al. [5] and they proved the following results:

- The star graph $K_{1,n}$ is square divisor cordial if and only if $n = 2, 3, 4, 5$ or 7.
- The complete bipartite graph $K_{2,n}$ is square divisor cordial.
- The complete bipartite graph $K_{3,n}$ is square divisor cordial if and only if $n = 1, 2, 3, 5, 6, 7$ or 9.
- The complete graph K_n is square divisor cordial if and only if $n = 1, 2, 3, 5$.

Vaidya and Shah [6] proved that:

- $B_{n,n}$ is a square divisor cordial graph.
- Restricted $B_{n,n}^2$ is a square divisor cordial graph.

We define cube divisor cordial labeling as follows.

Definition 1.5: Let $G = (V(G), E(G))$ be a simple graph and $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ be a bijection. For each edge $e = uv$, assign the label 1 if $[f(u)]^3 \mid f(v)$ or $[f(v)]^3 \mid f(u)$ and the label 0 otherwise. The function f is called a *cube divisor cordial labeling* if $|e_f(0) - e_f(1)| \leq 1$.

A graph which admits a cube divisor cordial labeling is called a *cube divisor cordial graph*.

Here we consider the following definitions of Standard Graphs.

- Bistar $B_{n,n}$ is the graph obtained by joining the center(apex) vertices of two copies of $K_{1,n}$ by an edge.
- For a simple connected graph G the square of graph G is denoted by G^2 and defined as the graph with the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 apart in G.

MAIN RESULTS

Theorem 2.1: The complete graph K_n is cube divisor cordial if and only if $n = 1, 2, 3, 4$.

Proof: Let $u_1, u_2, \ldots, u_n$ be the vertices of the complete graph K_n. We note that $|V(K_n)| = n$ and $|E(K_n)| = \frac{n(n-1)}{2}$.

Case 1: $n = 1, 2, 3, 4$.

In this case we define cube divisor cordial labeling $f : V(K_n) \rightarrow \{1, 2, \ldots, n\}$ as follows

$f(u_i) = i;\ i = 1, 2, 3, 4$.

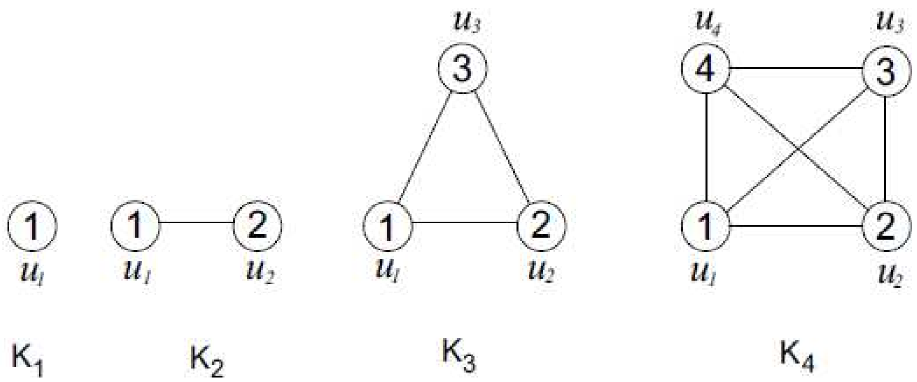

Figure 1: Cube Divisor Cordial Labeling of K_1, K_2, K_3, and K_4

Case 2: $n \geq 5$.

If possible, let there exist a cube divisor cordial labeling f of K_n.

Without loss of generality we assume that $f(u_i) = i;\ i = 1, 2, \ldots, n$.

Sub case 1: $n \equiv 0, 1 (mod\ 4)$.

Then obviously $e_f(i) = \frac{n(n-1)}{4};\ i = 0, 1$.

Since u_1 contributes $n-1$, u_2 contributes $\left\lfloor \frac{n}{2^3} \right\rfloor$, u_3 contributes $\left\lfloor \frac{n}{3^3} \right\rfloor$, too $e_f(1)$

We have

$$e_f(1) = n - 1 + \left\lfloor \frac{n}{2^3} \right\rfloor + \left\lfloor \frac{n}{3^3} \right\rfloor + \ldots + \left\lfloor \frac{n}{k^3} \right\rfloor$$

Where k is the largest positive integer such that $k^3 \leq n$.

Since $n \geq 5$

$$\frac{(n-1)(n-4)}{4} > \left\lfloor \frac{n}{2^3} \right\rfloor + \left\lfloor \frac{n}{3^3} \right\rfloor + \ldots + \left\lfloor \frac{n}{k^3} \right\rfloor$$

$$\frac{n(n-1)}{4}-(n-1)>\left\lfloor \frac{n}{2^3}\right\rfloor+\left\lfloor \frac{n}{3^3}\right\rfloor+\ldots+\left\lfloor \frac{n}{k^3}\right\rfloor$$

$$\frac{n(n-1)}{4}>(n-1)+\left\lfloor \frac{n}{2^3}\right\rfloor+\left\lfloor \frac{n}{3^3}\right\rfloor+\ldots+\left\lfloor \frac{n}{k^3}\right\rfloor$$

$$e_f(1)>(n-1)+\left\lfloor \frac{n}{2^3}\right\rfloor+\left\lfloor \frac{n}{3^3}\right\rfloor+\ldots+\left\lfloor \frac{n}{k^3}\right\rfloor$$

This is a contradiction**.**

Sub case 2: $n \equiv 2,3(mod\ 4)$.

Then obviously $e_f(1)=\left\lfloor \frac{n(n-1)}{4}\right\rfloor$ and $e_f(0)=\left\lceil \frac{n(n-1)}{4}\right\rceil$ or $e_f(1)=\left\lceil \frac{n(n-1)}{4}\right\rceil$ and $e_f(0)=\left\lfloor \frac{n(n-1)}{4}\right\rfloor$.

Since u_1 contributes $n-1$, u_2 contributes $\left\lfloor \frac{n}{2^3}\right\rfloor$, u_3 contributes $\left\lfloor \frac{n}{3^3}\right\rfloor$, to $e_f(1)$.

Let k be the largest positive integer such that $k^3 \le n$. We have

$$e_f(1)=n-1+\left\lfloor \frac{n}{2^3}\right\rfloor+\left\lfloor \frac{n}{3^3}\right\rfloor+\ldots+\left\lfloor \frac{n}{k^3}\right\rfloor.$$

$$<\left\lfloor \frac{n(n-1)}{4}\right\rfloor$$

$$<\left\lceil \frac{n(n-1)}{4}\right\rceil$$

This is a contradiction**.**

Hence, if $n \ge 5$ there cannot be a cube divisor cordial labeling.

Therefore, the complete graph K_n is cube divisor cordial if and only if $n=1,2,3,4$.

Theorem 2.2: The star graph $K_{1,n}$ is a cube divisor cordial graph if and only if $n=1,2,3$

Proof : Let u be the apex vertex and let $u_1,u_2,\ldots,u_n$ be the pendant vertices of the star $K_{1,n}$.

We note that $|V(K_{1,n})|=n+1$ and $|E(K_{1,n})|=n$. The proof of this result is divided into following two cases.

Case 1: $\mathrm{n}=1,2,3$**.**

In this case assign the label 2 to the vertex u and the remaining labels to the vertices $u_1,u_2,\ldots,u_n$.

If n is even then $e_f(0)=e_f(1)$ and if n is odd then $e_f(1)=e_f(0)+1$.

Thus, $|e_f(0)-e_f(1)|\le 1$. Hence, $K_{1,n}$ is a cube divisor cordial graph.

Case 2: $n \geq 4$**.**

Subcase 1: $n \equiv 0(mod\ 2)$.

In order to satisfy the edge condition of cube divisor cordial labeling we need $e_f(0) = e_f(1) = \frac{n}{2}$.

If we assign label 1 to apex vertex u then $e_f(1) = n$.

Therefore, $e_f(0) = 0$. Hence , $|e_f(0) - e_f(1)| = n \geq 4$.

Let $k \in \{2, 3, \ldots, n+1\}$.

If we assign label k to apex vertex u then we get at most $\left\lfloor \frac{n+1}{k^3} \right\rfloor + 1$ edges having label 1.

Therefore,

$$e_f(1) \leq \left\lfloor \frac{n+1}{k^3} \right\rfloor + 1$$

$$\leq \left\lfloor \frac{n+1}{8} \right\rfloor + 1$$

$$< \frac{n}{2}.$$

Hence, in any other labeling pattern we always get $e_f(1) < \frac{n}{2}$.

Thus, if n is even , there can not be a cube divisor cordial labeling.

Sub case 2: $n \equiv 1(mod\ 2)$.

In order to satisfy the edge condition of cube divisor cordial labeling we need either $e_f(0) = \frac{n-1}{2}$, $e_f(1) = \frac{n+1}{2}$ or $e_f(0) = \frac{n+1}{2}$, $e_f(1) = \frac{n-1}{2}$.

If we assign label 1 to apex vertex u then $e_f(1) = n$.

There fore, $e_f(0) = 0$. Hence, $|e_f(0) - e_f(1)| = n \geq 4$.

Let $k \in \{2, 3 \ldots, n+1\}$

If we assign label k to apex vertex u then we get at most $\left\lfloor \frac{n+1}{k^3} \right\rfloor + 1$ edges having label 1.

There fore,

$$e_f(1) \le \left\lfloor \frac{n+1}{k^3} \right\rfloor + 1$$

$$\le \left\lfloor \frac{n+1}{8} \right\rfloor + 1$$

$$< \frac{n-1}{2}$$

$$< \frac{n+1}{2}$$

Hence, in any other labeling pattern we always get $e_f(1) < \frac{n-1}{2} < \frac{n+1}{2}$

Thus, if n is odd, there can not be a cube divisor cordial labeling.

Therefore, the star graph $K_{1,n}$ is a cube divisor cordial graph if and only if $n = 1, 2, 3$.

Theorem 2.3: The complete bipartite graph $K_{2,n}$ is a cube divisor cordial graph.

Proof: Let $K_{2,n}$ be the complete bipartite graph. Let $W = U \cup V$ be the bipartition of $K_{2,n}$ such that $U = \{u_1, u_2\}$ and $V = \{v_1, v_2, \ldots, v_n\}$. We note that $|V(K_{2,n})| = n+2$ and $|E(K_{2,n})| = 2n$.

Now assign the label 1 to u_1, the largest prime number p such that $p \le n+2$ to u_2. Label the remaining vertices $v_1, v_2, \ldots, v_n$ from the set $\{2, 3, \ldots, p-1, p+1, \ldots, n+2\}$. Then if follows that $e_f(0) = e_f(1) = n$. Therefore, the complete bipartite graph $K_{2,n}$ is a cube divisor cordial graph.

Illustration 2.4. Cube divisor cordial labeling of the graph $K_{2,6}$ is shown in Figure 2.

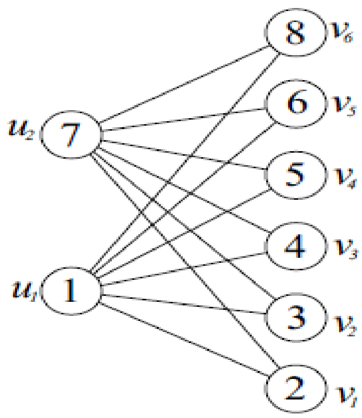

Figure 2: Cube Divisor Cordial Labeling Of $K_{2,6}$

Theorem 2.5: The complete bipartite graph $K_{3,n}$ is cube divisor cordial if and only if $n = 1, 2$.

Proof: Let $K_{3,n}$ be the complete bipartite graph. Let $W = U \cup V$ be the bipartition of $K_{3,n}$ such that

$U=\{u_1,u_2,u_3\}$ and $V=\{v_1,v_2,\ldots,v_n\}$. We note that $|V(K_{3,n})|=n+3$ and $|E(K_{3,n})|=3n$.

The proof of this result is divided into following two cases.

Case 1: $n=1,2$.

The cube divisor cordial labeling of $K_{3,1}$ is shown in following Figure 3.

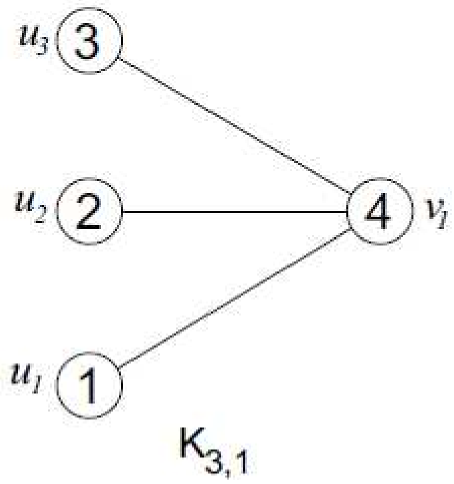

Figure 3: Cube Divisor Cordial Labeling of $K_{3,1}$

Here, we see that $e_f(1)=1$ and $e_f(0)=2$. Therefore, $|e_f(0)-e_f(1)|=1$.

The cube divisor cordial labeling of $K_{3,2}$ is shown in following Figure 4.

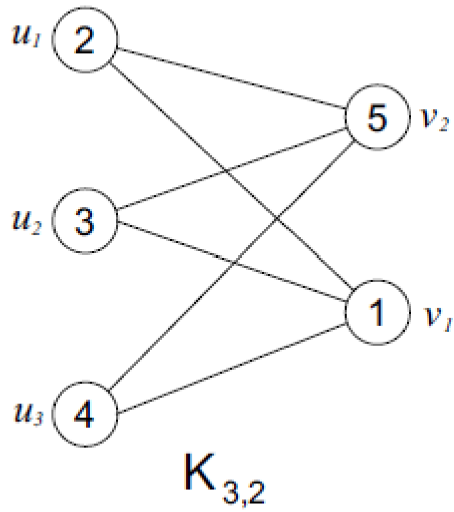

Figure 4: Cube Divisor Cordial Labeling of $K_{3,2}$

Here, we see that $e_f(1)=3$ and $e_f(0)=3$. Therefore, $|e_f(0)-e_f(1)|=0$.

Hence, $K_{3,1}$ and $K_{3,2}$ are cube divisor cordial graphs.

Case 2: $n \geq 3$.

Sub case 1: $n \equiv 0(mod\ 2)$.

In order to satisfy the edge condition of cube divisor cordial labeling we need $e_f(0) = e_f(1) = \frac{3n}{2}$.

If we assign label 1 to u_1, It contribuâtes at. Most n to $e_f(1)$, 2 to u_2, It contribuâtes at. Most $\left\lfloor \frac{n+3}{2^3} \right\rfloor$ to $e_f(1)$.

Let $k \in \{3, 4, \ldots, n+3\}$.

If we assign label k to u_3, It contribuâtes at. Most $\left\lfloor \frac{n+3}{k^3} \right\rfloor$ to $e_f(1)$.

There fore,

$$e_f(1) \leq n + \left\lfloor \frac{n+3}{2^3} \right\rfloor + \left\lfloor \frac{n+3}{k^3} \right\rfloor$$

$$\leq n + \left\lfloor \frac{n+3}{8} \right\rfloor + \left\lfloor \frac{n+3}{27} \right\rfloor$$

$$< \frac{3n}{2}$$

In any other labeling pattern we always get $e_f(1) < \frac{3n}{2}$.

Hence, there can not be a cube divisor cordial labeling.

Subcase 2: $n \equiv 1(mod\ 2)$.

In order to satisfy the edge condition of cube divisor cordial labeling we need $e_f(0) = \frac{3n+1}{2}$, $e_f(1) = \frac{3n-1}{2}$ or $e_f(0) = \frac{3n-1}{2}$, $e_f(1) = \frac{3n+1}{2}$.

If we assign label 1 to u_1, it contributes at most n to $e_f(1)$, 2 to u_2, it contributes at most $\left\lfloor \frac{n+3}{2^3} \right\rfloor$ to $e_f(1)$.

Let $k \in \{3, 4, \ldots, n+3\}$.

If we assign label k to u_3, it contributes at most $\left\lfloor \frac{n+3}{k^3} \right\rfloor$ to $e_f(1)$.

Therefore,

$$e_f(1) \leq n + \left\lfloor \frac{n+3}{2^3} \right\rfloor + \left\lfloor \frac{n+3}{k^3} \right\rfloor$$

$$\leq n + \left\lfloor \frac{n+3}{8} \right\rfloor + \left\lfloor \frac{n+3}{27} \right\rfloor$$

$$< \frac{3n-1}{2}$$

$$< \frac{3n+1}{2}$$

In any other labeling pattern we always get $e_f(1) < \frac{3n-1}{2} < \frac{3n+1}{2}$.

Hence, there can not be a cube divisor cordial labeling.

Therefore, the complete bipartite graph $K_{3,n}$ is cube divisor cordial if and only if $n = 1, 2$.

Theorem 2.6: The bistar $B_{n,n}$ is a cube divisor cordial graph.

Proof: Let $B_{n,n}$ be the bistar with vertex set $\{v, u, v_i, u_i; 1 \leq i \leq n\}$ where v_i, u_i are pendant vertices. We note that $|V(B_{n,n})| = 2n+2$ and $|E(B_{n,n})| = 2n+1$.

Define vertex labeling $f : V(B_{n,n}) \to \{1, 2, \dots, 2n+2\}$ as follows.

Let p be the largest prime number such that $p < 2n+2$.

$f(v) = 1$

$f(u) = p$

$f(v_i) = 2i$; $1 \leq i \leq n$

Label the remaining vertices $u_1, u_2, \dots, u_n$ from the set $\{3, 5, 7, \dots, p-2, p+2, \dots, 2n-1, 2n+1, 2n+2\}$.

In view of the above labeling pattern we have $e_f(0) = n$ and $e_f(1) = n+1$.

Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence, the bistar $B_{n,n}$ is a cube divisor cordial graph.

Illustration 2.7: Cube divisor cordial labeling of the graph $B_{7,7}$ is shown in Figure 5.

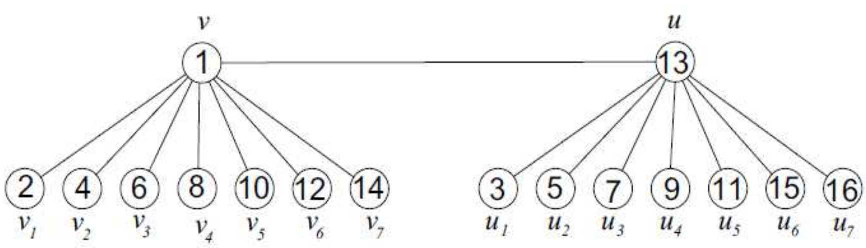

Figure 5: Cube Divisor Cordial Labeling of $B_{7,7}$

Theorem 2.8: Restricted $B^2_{n,n}$ is a cube divisor cordial graph.

Proof: Consider $B_{n,n}$ with vertex set $\{v,u,v_i,u_i;1\le i\le n\}$, where v_i,u_i are pendant vertices. Let G be the restricted $B^2_{n,n}$ graph with $V(G)=V(B_{n,n})$ and $E(G)=E(B_{n,n})\cup\{uv_i,vu_i/1\le i\le n\}$ then $|V(G)|=2n+2$ and $|E(G)|=4n+1$.

Define vertex labeling $f:V(G)\to\{1,2,\ldots,2n+2\}$ as follows.

Let p be the largest prime number such that $p<2n+2$.

$f(v)=1$

$f(u)=p$

$f(v_i)=2i$; $1\le i\le n$

Label the remaining vertices $u_1,u_2,\ldots,u_n$ from the set $\{3,5,7,\ldots,p-2,p+2,\ldots,2n-1,2n+1,2n+2\}$.

In view of the above labeling pattern we have $e_f(0)=2n$ and $e_f(1)=2n+1$.

Thus, $|e_f(0)-e_f(1)|\le 1$.

Hence, restricted $B^2_{n,n}$ is a cube divisor cordial graph.

Illustration 2.9. Cube divisor cordial labeling of the graph restricted $B^2_{5,5}$ is shown in Figure 6.

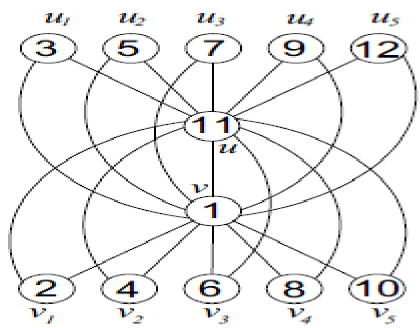

Figure 6: Cube Divisor Cordial Labeling of Restricted $B^2_{5,5}$

CONCLUDING REMARKS

To derive any result in the theory of cube divisor cordial labeling except above six results is an open problem.

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